



THE UNIVERSITY *of* EDINBURGH

## Edinburgh Research Explorer

# Long time scaling behaviour for diffusion with resetting and memory

### Citation for published version:

Boyer, D, Evans, MR & Majumdar, SN 2017, 'Long time scaling behaviour for diffusion with resetting and memory', *Journal of Statistical Mechanics: Theory and Experiment*. <https://doi.org/10.1088/1742-5468/aa58b6>

### Digital Object Identifier (DOI):

[10.1088/1742-5468/aa58b6](https://doi.org/10.1088/1742-5468/aa58b6)

### Link:

[Link to publication record in Edinburgh Research Explorer](#)

### Document Version:

Peer reviewed version

### Published In:

Journal of Statistical Mechanics: Theory and Experiment

### General rights

Copyright for the publications made accessible via the Edinburgh Research Explorer is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

### Take down policy

The University of Edinburgh has made every reasonable effort to ensure that Edinburgh Research Explorer content complies with UK legislation. If you believe that the public display of this file breaches copyright please contact [openaccess@ed.ac.uk](mailto:openaccess@ed.ac.uk) providing details, and we will remove access to the work immediately and investigate your claim.



# Long time scaling behaviour for diffusion with resetting and memory

Denis Boyer\*

*Instituto de Física, Universidad Nacional Autónoma de México, D.F. 04510, México and  
Centro de Ciencias de la Complejidad, Universidad Nacional Autónoma de México, D.F. 04510, México*

Martin R. Evans†

*SUPA, School of Physics and Astronomy, University of Edinburgh,  
Peter Guthrie Tait Road, Edinburgh EH9 3FD, United Kingdom*

Satya N. Majumdar‡

*Laboratoire de Physique Théorique et Modèles Statistiques, UMR 8626,  
Université Paris Sud 11 and CNRS, Bât. 100, Orsay F-91405, France*

We consider a continuous-space and continuous-time diffusion process under resetting with memory. A particle resets to a position chosen from its trajectory in the past according to a memory kernel. Depending on the form of the memory kernel, we show analytically how different asymptotic behaviours of the variance of the particle position emerge at long times. These range from standard diffusive ( $\sigma^2 \sim t$ ) all the way to anomalous ultraslow growth  $\sigma^2 \sim \ln \ln t$ .

PACS numbers: 05.40.-a, 02.50.-r

## I. INTRODUCTION

Diffusion is a fundamental dynamical process, introduced originally to describe the motion of molecules and Brownian motion [1] and subsequently used to model a whole range of stochastic phenomena from biology to computer science [2, 3]. The molecular origin of diffusion is the random walk of the constituent particles and one obtains the diffusion equation as a continuum limit of the random walk. The solution of the diffusion equation leads to the Gaussian distribution with variance growing linearly in time  $\sigma^2(t) \sim t$  which we refer to as diffusive growth. In an infinite system the Gaussian distribution keeps broadening in time, whereas in a finite system an equilibrium stationary state is attained at long times.

While diffusive processes are abundant in nature, many systems exhibit anomalous behaviour  $\sigma^2(t) \sim t^{\beta_1}$  where  $\beta_1 \neq 1$ . For example,  $\beta_1 < 1$  corresponds to subdiffusion whereas  $\beta_1 > 1$  corresponds to superdiffusion [4, 5]. Models such as the continuous time random walk (CTRW) provides a microscopic model for subdiffusive behaviour [6]. In contrast Lévy flights or walks lead to superdiffusive behaviour [4, 5]. Another anomalous behaviour, outside the regime of these models, is the very slow logarithmic growth with time  $\sigma^2 \sim (\ln t)^{\beta_2}$  where  $\beta_2 > 0$ . Such logarithmically slow dynamics is usually observed in glassy or disordered systems. For example in the Sinai model—a single random walker diffusing in a Brownian random potential—the variance of the position of the particle grows in time as  $\sigma^2 \sim (\ln t)^4$  [4]. A natural question is whether microscopic dynamics can generate such anomalously slow growth of the variance of a single particle without the explicit presence of disorder or external potential.

Recent studies have revealed that a variety of subdiffusive behaviour, including logarithmic growth can be achieved in a single particle model with memory-dependent dynamics. In this simple scenario the particle, at a given time, jumps a random distance which may depend on the history of the process; these processes are thus typically non-Markovian in nature. Examples include ‘Elephant Walk’ [7], ‘Cookie Random Walks’ [8, 9], ‘Reinforced Walks’ [10, 11], etc. In the ecological context, a model [12, 13] of animal mobility has been proposed which incorporates the tendency for an animal to revisit previously visited sites, a possible mechanism that can explain the formation of home ranges. A particular case of this model is the one where, in addition to nearest neighbour random walk dynamics, the particle revisits a site with a probability proportional to the number of times this site has been visited in the past [14]. This model turns out to be exactly solvable [14] and the particle position distribution converges to a Gaussian at long times with a variance growing logarithmically as  $\sigma^2 \sim \ln t$ . The model was generalised in [15] to relocation to a site visited in the past, selected according to a time-weighted distribution with a specific form. This has been further

---

\*Electronic address: boyer@fisica.unam.mx

†Electronic address: mevans@staffmail.ed.ac.uk

‡Electronic address: majumdar@lptms.u-psud.fr

generalised to memory kernels that lead to an asymptotic Lévy distribution with index  $0 < \mu \leq 2$  for the particle position with a length scale growing as  $(\ln t)^{1/\mu}$  [16]. Another simple model with memory that has attracted recent interest is diffusion with stochastic resetting [17, 18]. In this case the dynamics occurs in continuous space-time and, in addition to diffusion, relocation occurs to the initial position with a constant rate. In this case at long times the particle position distribution converges to a non-Gaussian stationary distribution and the variance approaches a constant at late times [18, 19]. Several non-Markovian variations of this model have been studied recently [20–24].

These studies demonstrate how memory can strongly affect the late time behaviour of a diffusive process and generate a range of time dependent growth of the variance of the particle position. It is then natural to ask whether a simple model can incorporate all the different behaviours, found in the various contexts mentioned above, within a single unified setting. The purpose of this paper is to present and analyse such a generalized model.

In the present work we consider diffusion (in continuous space and time) with a resetting process that relocates the diffusive particle to a position from its past history. The model allows a smooth interpolation between diffusion with resetting to the initial position and diffusion with time-weighted relocation to positions visited in the past. We demonstrate conditions required for the diffusive particle to attain a localised, nonequilibrium stationary state. Furthermore, we demonstrate that when the conditions for a stationary state do not hold, a wider variety of long-time scaling behaviours than previously encountered are possible: the variance of the distribution may grow logarithmically in time, subdiffusively as  $t^\beta$  with  $\beta < 1$  or diffusively as  $t$  with a modified or unmodified diffusion constant. In addition we find for a special case, at the localised-nonlocalised transition, an ultra-slow  $\ln \ln t$  growth of the variance. We note that a similar ultraslow spread of a spatial distribution has been reported from data on human mobility [25].

## II. THE MODEL

We consider a continuous-space and continuous-time implementation of diffusion under resetting with memory. We begin by considering spatial dimension  $d = 1$  but our results are easily generalisable to arbitrary  $d$  (see Section VI). We define the resetting rate to be  $r$  and upon resetting at time  $t$  the diffusive particle selects a position  $x(\tau)$  from its trajectory, where  $\tau$  is chosen with probability density  $K(\tau, t)$ . We refer to  $K(\tau, t)$  as the memory kernel, which can be any non-negative function and has the property that  $\int_0^t d\tau K(\tau, t) = 1$  for all  $t$ . This property guarantees the conservation of total probability. Figure 1 illustrates these rules with a diffusive trajectory undergoing a first resetting event to a position occupied at a former time.

The Master equation for the process reduces to the Fokker-Planck equation for the probability distribution  $p(x, t)$

$$\frac{\partial p(x, t)}{\partial t} = D \frac{\partial^2 p(x, t)}{\partial x^2} - r p(x, t) + r \int_0^t K(\tau, t) p(x, \tau) d\tau \quad (1)$$

starting from the initial condition  $p(x, 0) = \delta(x)$ . The first term on the right hand side of (1) represents diffusion with diffusion constant  $D$ ; the second term represents loss of probability from  $x$  due to resetting; the third term represents gain of probability into  $x$  by choosing a time  $\tau$  in the past with probability density  $K(\tau, t)$  and relocating to  $x$  with probability density  $p(x, \tau)$ . Indeed, by integrating Eq. (1) over  $x$  and using  $\int_0^t d\tau K(\tau, t) = 1$ , it is easy to see that the Master equation conserves the total probability  $\int dx p(x, t) = 1$  since the integral of the second term is  $-r$  and the integral of the third term is  $+r$ . Equation (1) may be derived following a similar approach to that used for the discrete time random walk case [15].

Let us first consider two limiting cases of the kernel  $K(\tau, t)$ :

(i) **Case I:**  $K(\tau, t) = \delta(\tau)$ . In this case, Eq. (1), upon using  $p(x, 0) = \delta(x)$ , reduces to

$$\frac{\partial p(x, t)}{\partial t} = D \frac{\partial^2 p(x, t)}{\partial x^2} - r p(x, t) + r \delta(x). \quad (2)$$

This then corresponds to resetting the walker to its initial position  $x_0 = 0$  with rate  $r$ . Hence in this case, the model reduces to the ‘diffusion with stochastic resetting’ model studied by Evans and Majumdar [18]. For that model the probability distribution reaches a localised, stationary state exponentially decreasing with distance from the resetting position  $p_{\text{st}}(x) = \frac{\alpha_0}{2} \exp[-\alpha_0 |x|]$  with  $\alpha_0 = (r/D)^{1/2}$  [18].

(ii) **Case II:**  $K(\tau, t) = 1/t$ . In this case, the Master equation becomes

$$\frac{\partial p(x, t)}{\partial t} = D \frac{\partial^2 p(x, t)}{\partial x^2} - r p(x, t) + \frac{r}{t} \int_0^t p(x, \tau) d\tau. \quad (3)$$

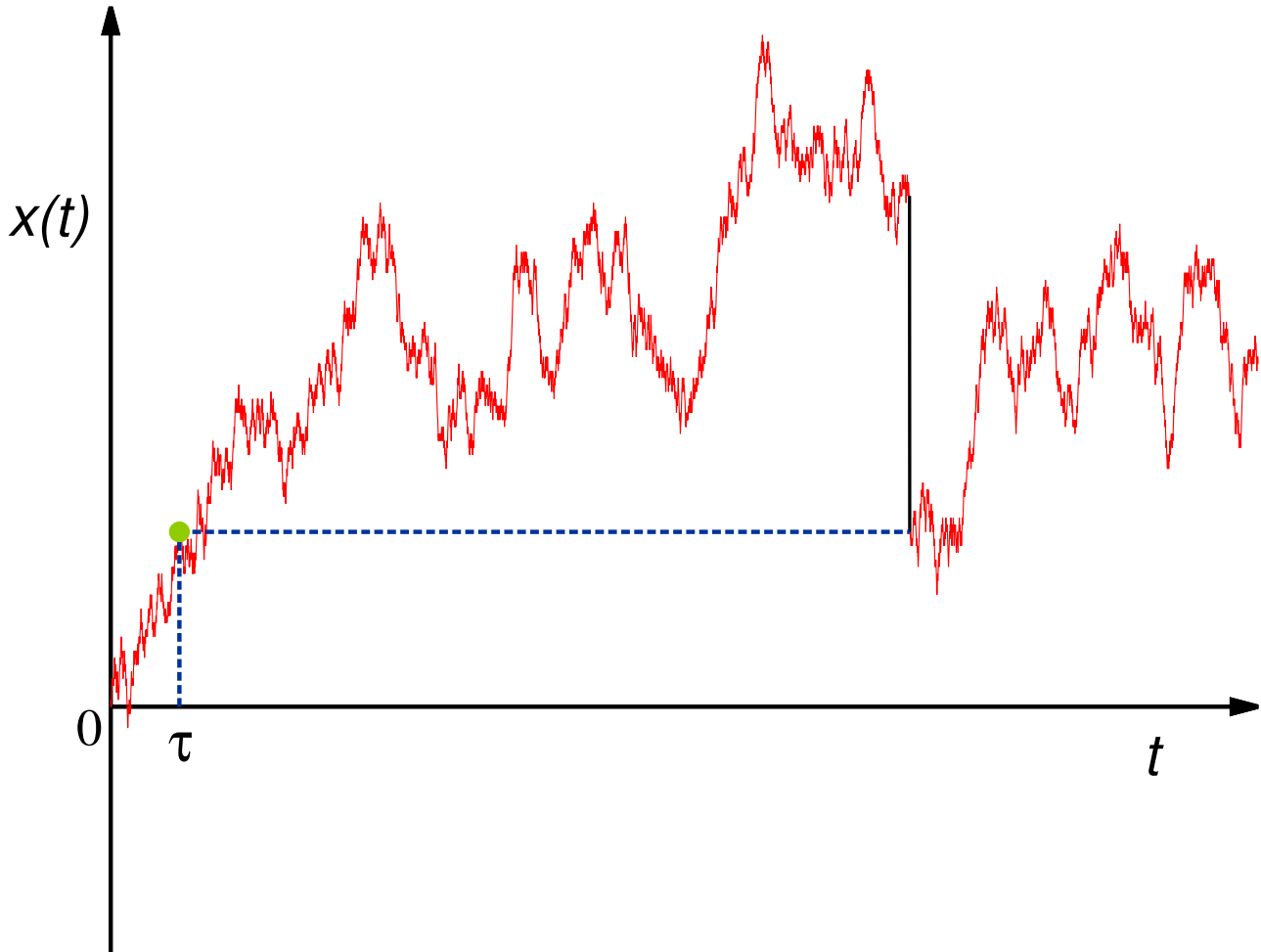


FIG. 1: (Color online) Example of a trajectory starting at the origin and following the rules of the model considered in this study. At fixed rate  $r$ , the normally diffusive particle resets to a previous position  $x(\tau)$  where the time  $\tau$  is a random variable drawn from a given distribution  $K(\tau, t)$ .

This is just the continuous-space and continuous-time version of the discrete-time lattice model studied by Boyer and Solis-Salas [14], where the walker resets with rate  $r$  to a previously visited site chosen randomly from all past times. This is equivalent to saying that a previously visited site is chosen for relocation with a probability proportional to the number of visits to that site in the past. In that work it was shown that the late time behaviour is Gaussian but with a variance growing very slowly (logarithmically) with time [14]. We will show in Section IV that, as in the corresponding random walk problem, the late time behaviour of Eq. (3) is Gaussian (with zero mean):  $p(x, t) \sim \exp[-x^2/2\sigma^2(t)]$  with a variance growing very slowly (logarithmically) with time [14]:  $\sigma^2(t) \sim \ln t$ .

Our interest is in establishing what further behaviours are possible under resetting with memory. To this end, we choose a family of memory kernels  $K(\tau, t)$  which allows a smooth interpolation between these two limiting cases:

$$K(\tau, t) = \frac{\phi(\tau)}{\int_0^t \phi(\tau) d\tau} \quad (4)$$

where  $\phi(\tau)$  can be any non-negative function. Thus  $K(\tau, t)$  only depends on the present time  $t$  through the denominator in (4). By choosing  $\phi(\tau) = \delta(\tau)$  one obtains Case I, where  $p(x, t)$  approaches a non-Gaussian stationary state at late times. In contrast, by choosing  $\phi(\tau) = 1$  one recovers case II where the distribution is a Gaussian at late times with variance growing logarithmically with time  $t$ . We would like to understand generally how the late time distribution depends on the choice of  $\phi(\tau)$ .

We note that another class of memory kernel,

$$K(\tau, t) = \frac{\psi(t - \tau)}{\int_0^t \psi(t - \tau) d\tau} \quad (5)$$

was considered in the context of random walks with relocation and the special case  $\psi(t - \tau) = (1 + t - \tau)^{-\beta}$  (memory decaying as a power law with exponent  $\beta > 0$ ) was analysed in detail [15]. Except in the interval  $1 < \beta < 2$ , this choice led to late time Gaussian distributions  $p(x, t) \sim \exp[-x^2/2\sigma^2(t)]$ , but now the temporal growth of the variance  $\sigma^2(t)$  depends on the choice of the exponent  $\beta$ . It was found that  $\sigma^2(t) \sim t$  for  $\beta > 2$  and  $\sigma^2(t) \sim \ln t$  for  $\beta < 1$  [15]. In the intermediate case  $1 < \beta < 2$ ,  $p(x, t)$  is non-Gaussian and known only through its moments, and  $\sigma^2(t) \sim t^{\beta-1}$ .

The Fokker-Planck equation (1) is linear in  $p(x, t)$ , and hence it is natural to consider the Fourier transform of  $p(x, t)$

$$\tilde{p}(k, t) = \int_{-\infty}^{\infty} p(x, t) e^{ikx} dx. \quad (6)$$

Taking the Fourier transform of Eq. (1), with the choice for kernel in (4), yields

$$\frac{\partial \tilde{p}(k, t)}{\partial t} = -(r + D k^2) \tilde{p}(k, t) + r \frac{\int_0^t \phi(\tau) \tilde{p}(k, \tau) d\tau}{\int_0^t \phi(\tau) d\tau}, \quad (7)$$

subject to the initial condition,  $\tilde{p}(k, 0) = 1$ . In the next two sections we shall consider cases where (7) can be solved exactly. Then in Section V we shall extract the large time behaviour of the solution in the general case.

### III. EXACT SOLUTION FOR THE CASE $\phi(\tau) = \lambda \exp[-\lambda \tau]$

Let us first consider the special case when

$$\phi(\tau) = \lambda \exp[-\lambda \tau]. \quad (8)$$

When  $\lambda \rightarrow \infty$ ,  $\phi(\tau) \rightarrow \delta(\tau)$  and we recover Case I. In the opposite limit  $\lambda \rightarrow 0$ ,  $K(\tau, t) = 1/t$  and we recover Case II. However as the limits  $\lambda \rightarrow 0$  and  $t \rightarrow \infty$  do not commute we shall consider the case  $\lambda = 0$  separately in the next section.

Substituting  $\phi(\tau) = \lambda \exp[-\lambda \tau]$  in Eq. (7) gives

$$(1 - e^{-\lambda t}) \left[ \frac{\partial \tilde{p}(k, t)}{\partial t} + (r + D k^2) \tilde{p}(k, t) \right] = r \lambda \int_0^t \tilde{p}(k, \tau) e^{-\lambda \tau} d\tau. \quad (9)$$

Differentiating once more with respect to  $t$  reduces (9) to a second order differential equation

$$(e^{\lambda t} - 1) \left[ \frac{\partial^2 \tilde{p}(k, t)}{\partial t^2} + (r + D k^2) \frac{\partial \tilde{p}(k, t)}{\partial t} \right] + \lambda \left[ \frac{\partial \tilde{p}(k, t)}{\partial t} + D k^2 \tilde{p}(k, t) \right] = 0. \quad (10)$$

To solve this equation, we first make a change of variable, from  $t$  to  $y = e^{-\lambda t}$ . Then,

$$\tilde{p}(k, t) = W(y = e^{-\lambda t}) \quad (11)$$

where  $W(y)$  depends implicitly on  $k$ . Substituting (11) in Eq. (10), we get after a few steps of straightforward algebra, a second order differential equation for  $W(y)$

$$y(1 - y) \frac{d^2 W(y)}{dy^2} + \left[ 1 - \frac{1}{\lambda} (r + D k^2) - \left( 2 - \frac{r + D k^2}{\lambda} \right) y \right] \frac{dW(y)}{dy} + \frac{D k^2}{\lambda} W(y) = 0. \quad (12)$$

As  $0 \leq e^{-\lambda t} \leq 1$ , the second order differential equation (12) holds in the regime  $0 \leq y \leq 1$ . The initial condition,  $\tilde{p}(k, t = 0) = 1$  and the fact that  $y = e^{-\lambda t} \rightarrow 1$  as  $t \rightarrow 0$ , translates into one boundary condition for  $W(y)$

$$W(y = 1) = 1. \quad (13)$$

To solve a second order equation, we generally need two boundary conditions. However, we note that the original equation (9) (before taking the derivative with respect to time) is a first order differential equation. Hence, it turns out that just one initial condition in  $t$  (or equivalently one boundary condition at  $y = 1$ ,  $W(1) = 1$ ) is enough to fix the full solution of (12).

Fortunately, Eq. (12) has the form of a standard hypergeometric differential equation

$$y(1-y)\frac{d^2W(y)}{dy^2} + [c - (a+b+1)y]\frac{dW(y)}{dy} - abW(y) = 0, \quad (14)$$

once we identify

$$a = \frac{1}{2} \left[ c + \sqrt{c^2 + \frac{4Dk^2}{\lambda}} \right] \quad (15)$$

$$b = \frac{1}{2} \left[ c - \sqrt{c^2 + \frac{4Dk^2}{\lambda}} \right] \quad (16)$$

$$c = 1 - \frac{r + Dk^2}{\lambda} = a + b. \quad (17)$$

The general solution of the hypergeometric differential equation (14) can be written as a linear combination [26]

$$W(y) = AF(a, b, c; y) + By^{1-c}F(a-c+1, b-c+1, 2-c; y) \quad 0 \leq y \leq 1, \quad (18)$$

where  $F(a, b, c; y)$  is the standard hypergeometric function and  $A$  and  $B$  are two arbitrary constants, yet to be determined. To fix the two constants  $A$  and  $B$ , we will use the boundary condition  $W(y=1) = 1$  in Eq. (13). In order to do so, it is convenient to use the following identity [26]

$$F(a, b, a+b; y) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} [2\psi(n+1) - \psi(a+n) - \psi(b+n) - \ln(1-y)] (1-y)^n \quad (19)$$

where  $(a)_n = a(a+1)(a+2)\dots(a+n-1)$  is the Pochhammer symbol and  $\psi(z) = \Gamma'(z)/\Gamma(z)$ . As  $y \rightarrow 1$ , keeping only the leading  $n=0$  term gives

$$F(a, b, a+b; y \rightarrow 1) \rightarrow \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} [2\psi(1) - \psi(a) - \psi(b) - \ln(1-y)]. \quad (20)$$

Using this result in Eq. (18) gives

$$\begin{aligned} W(y \rightarrow 1) &= A \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} [2\psi(1) - \psi(a) - \psi(b) - \ln(1-y)] \\ &\quad + B \frac{\Gamma(2-a-b)}{\Gamma(1-a)\Gamma(1-b)} [2\psi(1) - \psi(1-a) - \psi(1-b) - \ln(1-y)]. \end{aligned} \quad (21)$$

In order to enforce the boundary condition  $W(y=1) = 1$  the coefficient of the leading diverging term  $\ln(1-y)$  in Eq. (21) must vanish. In addition, the subleading constant term must be equal to unity. This then provides two equations to determine the two unknown constants  $A$  and  $B$ . These two equations read

$$A \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} + B \frac{\Gamma(2-a-b)}{\Gamma(1-a)\Gamma(1-b)} = 0 \quad (22)$$

$$A \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} [2\psi(1) - \psi(a) - \psi(b)] + B \frac{\Gamma(2-a-b)}{\Gamma(1-a)\Gamma(1-b)} [2\psi(1) - \psi(1-a) - \psi(1-b)] = 1. \quad (23)$$

Using (22) in (23) gives

$$A \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} [\psi(1-a) + \psi(1-b) - \psi(a) - \psi(b)] = 1. \quad (24)$$

This determines  $A$ . The other constant  $B$  can be determined using (22). The expressions for  $A$  and  $B$  can be further simplified by using the relation [26],  $\psi(1-a) - \psi(a) = \pi \cot(\pi a)$  and the two identities:  $\cot(\pi a) + \cot(\pi b) = \sin(\pi(a+b))/[\sin(\pi a)\sin(\pi b)]$  and  $\Gamma(a)\Gamma(1-a) = \pi/\sin(\pi a)$  which eventually lead to

$$A = \frac{\Gamma(1-a-b)}{\Gamma(1-a)\Gamma(1-b)} \quad (25)$$

$$B = -\frac{1}{1-a-b} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}. \quad (26)$$

Plugging these expressions into (21) and equating  $W(y) = \tilde{p}(k, t)$  gives the exact time-dependent solution at all times  $t$

$$\begin{aligned} \tilde{p}(k, t) = & \frac{\Gamma(1-a-b)}{\Gamma(1-a)\Gamma(1-b)} F(a, b, a+b; e^{-\lambda t}) \\ & - \frac{1}{1-a-b} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} e^{-\lambda(1-c)t} F(1-b, 1-a, 2-a-b; e^{-\lambda t}), \end{aligned} \quad (27)$$

where, we recall that  $a$  and  $b$  are given in Eqs. (15) and (16).

It is evident from this exact solution in Eq. (27) that for the choice,  $\phi(\tau) = \lambda \exp[-\lambda\tau]$  with  $\lambda > 0$ , the system always reaches a stationary state as  $t \rightarrow \infty$ . Taking  $t \rightarrow \infty$  in Eq. (27), using  $F(a, b, c; y = 0) = 1$  and  $c < 1$ , one obtains exactly the stationary probability distribution (in Fourier space)

$$\tilde{p}_{\text{st}}(k) = \frac{\Gamma(1-a-b)}{\Gamma(1-a)\Gamma(1-b)}, \quad (28)$$

with  $a, b$  given in Eqs. (15) and (16). We can rewrite expression (28) as

$$\tilde{p}_{\text{st}}(k) = B(1-a-b, a) \frac{\sin \pi a}{\pi} \quad (29)$$

where  $B(x, y)$  is the usual Beta function [26].

Let us first verify that in the limit  $\lambda \rightarrow \infty$  limit, we recover the results of Ref. [18]. As  $\lambda \rightarrow \infty$ , we get from Eqs. (15), (16) and (17)

$$c = 1 - \frac{r + D k^2}{\lambda} \quad (30)$$

$$a \rightarrow 1 - \frac{r}{\lambda} \quad (31)$$

$$b \rightarrow -\frac{D k^2}{\lambda}. \quad (32)$$

Using  $\Gamma(x) \rightarrow 1/x$  as  $x \rightarrow 0$  in (28), gives

$$\tilde{p}_{\text{st}}(k) \xrightarrow{\lambda \rightarrow \infty} \frac{r}{r + D k^2}. \quad (33)$$

Inverting the Fourier transform exactly reproduces the stationary solution in real space obtained in Ref. [18]

$$p_{\text{st}}(x) = \sqrt{\frac{r}{4D}} \exp \left[ -\sqrt{\frac{r}{D}} |x| \right]. \quad (34)$$

For finite  $\lambda$ , one notices that the right hand side of Eq. (28) (or equivalently Eq. (29)) has poles when  $1-a-b = 1-c = 1 - (r + D k^2)/\lambda = -m$  where  $m = 0, 1, 2, \dots$ . In other words, the poles in the complex  $k$  plane occur at  $k = \pm i \sqrt{(r + \lambda m)/D}$ . Hence, the solution in the real space can be written as a sum over exponentials (each coming from a pole in the lower half of the complex  $k$ -plane for  $x > 0$  and from the upper half if  $x < 0$ )

$$p_{\text{st}}(x) = \sum_{m=0}^{\infty} A_m e^{-\sqrt{\frac{r+\lambda m}{D}} |x|}, \quad (35)$$

Thus at finite  $\lambda > 0$  the leading term of  $p_{\text{st}}(x)$  at large  $|x|$  is still given by  $e^{-\sqrt{r/D}|x|}$ . The coefficients  $A_m$ 's can be computed from the residues at the relevant poles and one obtains

$$A_m = \frac{\lambda}{2\pi D^{1/2}} \frac{(1-a_m)_m}{m!(r+m\lambda)^{1/2}} \sin \pi a_m \quad (36)$$

where

$$a_m = \frac{1}{2} \left[ 1 + m + ((1-m)^2 - 4r/\lambda)^{1/2} \right]. \quad (37)$$

#### IV. EXACT SOLUTION FOR UNIFORM MEMORY KERNEL: $\phi(\tau) = 1$

In the previous section we considered  $\phi(\tau)$  given by (8) for  $\lambda > 0$ . Since the asymptotic results only hold for  $\lambda > 0$ , we need to consider the special case  $\lambda \rightarrow 0$  (which corresponds to  $\phi(\tau) = 1$ ) separately, as presented below. This case actually corresponds to the continuous space-time limit of [14].

We consider  $\phi(\tau) = 1$  or  $K(t, \tau) = 1/t$  in which case we obtain from (1)

$$t \frac{\partial^2 \tilde{p}(k, t)}{\partial t^2} + (1 + (r + Dk^2)t) \frac{\partial \tilde{p}(k, t)}{\partial t} + Dk^2 \tilde{p}(k, t) = 0, \quad (38)$$

an equation that can be also deduced from the  $\lambda \rightarrow 0$  limit of (10).

We now define  $u = Dk^2 + r$  and  $v = Dk^2$  in (38). Then

$$t \frac{\partial^2 \tilde{p}(k, t)}{\partial t^2} + (1 + ut) \frac{\partial \tilde{p}(k, t)}{\partial t} + v \tilde{p}(k, t) = 0.$$

Making a further substitution  $z = -ut$  gives

$$z \frac{\partial^2 \tilde{p}(k, z)}{\partial z^2} + (1 - z) \frac{\partial \tilde{p}(k, z)}{\partial z} - \frac{v}{u} \tilde{p}(k, z) = 0 \quad (39)$$

which is Kummer's equation

$$z \frac{\partial^2 w(z)}{\partial z^2} + (b - z) \frac{\partial w(z)}{\partial z} - aw(z) = 0 \quad (40)$$

with  $b = 1$ ,  $a = \frac{v}{u}$ . We note that Kummer's equation (39) is a degenerate case of the hypergeometric equation (14) and the Kummer functions are confluent hypergeometric functions. For all values of  $a$  and  $z$ , and  $b$  not a non-positive integer, the Kummer functions are given by

$$M(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!}, \quad (41)$$

where  $(a)_n$  is again the Pochhammer symbol. As the other solution to (39) diverges we deduce  $\tilde{p}(k, t) = C(k)M\left(\frac{v}{u}, 1, -ut\right)$  where  $C(k)$  is to be fixed. From the initial condition  $x_0 = 0$  and the fact that in the limit  $z \rightarrow 0$ ,  $M(a, b, z) \rightarrow 1$  we find  $C(k) = 1$ .

In principle the exact probability density is obtained by inverting the inverse Fourier transform,

$$p(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{-ikx} \sum_{n=0}^{\infty} \left( \frac{Dk^2}{Dk^2 + r} \right)_n \frac{(-(Dk^2 + r)t)^n}{n! n!} \quad (42)$$

but the inversion is difficult to carry out.

However, we can easily compute the variance of the distribution (mean squared displacement), exact for all times, from the Fourier transform using the formula

$$\langle x^2 \rangle = - \left. \frac{\partial^2 \tilde{p}(k, t)}{\partial k^2} \right|_{k=0}. \quad (43)$$

Now we find

$$\left. \frac{\partial^2}{\partial k^2} (a)_n \right|_{k=0} = \frac{\partial^2}{\partial k^2} \left[ \left( \frac{Dk^2}{Dk^2 + r} \right)_n \right] \Big|_{k=0} = \frac{2D}{r} (n-1)!, \quad (44)$$

so that we obtain

$$\langle x^2 \rangle = \frac{2D}{r} \sum_{n=1}^{\infty} \frac{(rt)^n (-1)^{n+1}}{n! n}. \quad (45)$$

The asymptotic, large  $t$  form of the second moment can be obtained by noting

$$\langle x^2 \rangle = \frac{2D}{r} [E_1(rt) + \gamma + \ln(rt)], \quad (46)$$



where the exponential integral  $E_1(z)$  is given by

$$E_1(z) = \int_z^\infty dx \frac{e^{-x}}{x} = -\gamma - \ln z - \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n!n} \quad (47)$$

and  $\gamma = 0.5772\dots$  is Euler's constant. Substituting the asymptotic expansion of  $E_1(z)$

$$E_1(z) \sim \frac{e^{-z}}{z} \left( 1 - O\left(\frac{1}{z}\right) \right) \quad \text{for } z \rightarrow \infty, \quad (48)$$

into (46), we obtain

$$\langle x^2 \rangle = \frac{2D}{r} \left( \ln(rt) + \gamma + \frac{e^{-rt}}{rt} + \dots \right) \quad \text{for } t \gg 1. \quad (49)$$

This result is consistent with the that of [15] for the discrete, random walk case.

## V. GENERAL MEMORY KERNEL

We now consider a general memory kernel given by a general function  $\phi(\tau)$ . It is possible to analyse Eq. (7) at late times  $t$  according to the large  $\tau$  behaviour of  $\phi(\tau)$ . We find a rather rich variety of asymptotic behaviour of  $\sigma^2(t)$  which we summarize below.

- When  $\phi(\tau)$  decays more quickly than  $1/\tau$  for large  $\tau$  i.e.  $\tau\phi(\tau) \rightarrow 0$  as  $\tau \rightarrow \infty$  there is a stationary distribution  $p_{st}(x)$ .
- When  $\phi(\tau)$  increases, or decays as, or more slowly than,  $1/\tau$  for large  $\tau$ , i.e.,  $\tau\phi(\tau) > 0$  as  $\tau \rightarrow \infty$ , there is no stationary distribution. Instead there is a late time behaviour in which the time-dependent distribution takes a Gaussian form with variance  $\sigma^2(t)$ .

The time dependence of the variance has different classes of behaviour as follows:

1. for  $\phi(\tau) \sim 1/\tau$ ,  $\sigma^2(t) \sim \ln \ln t$ ;
2. for  $\phi(\tau) \sim \tau^\alpha$  with  $\alpha > -1$ ,  $\sigma^2(t) \sim \ln t$ ;
3. for  $\phi(\tau) \sim \exp(a\tau^\beta)$  where  $0 < \beta < 1$  and  $a$  is a positive constant,  $\sigma^2(t) \sim t^\beta$ ;
4. for  $\phi(\tau) \sim \exp(a\tau)$  where  $a$  is a positive constant,  $\sigma^2(t) \simeq \left( \frac{2a}{a+r} \right) Dt$ ;
5. for  $\phi(\tau) \sim \exp(a\tau^\beta)$  where  $\beta > 1$  and  $a$  is a positive constant,  $\sigma^2(t) \simeq 2Dt$ .

We now discuss these different cases separately.

### A. $\phi(\tau)$ decaying faster than $1/\tau$ as $\tau \rightarrow \infty$

Consider the case where  $\phi(\tau) \ll 1/\tau$  as  $\tau \rightarrow \infty$ . In this case, the integral  $\int_0^t \phi(\tau) d\tau$  appearing in the denominator of the second term in Eq. (7) converges as  $t \rightarrow \infty$ . Hence, we expect that as  $t \rightarrow \infty$ , the system approaches a stationary state, and the stationary solution (in Fourier space) is obtained by setting the rhs of Eq. (7) to zero

$$(r + Dk^2)\tilde{p}_{st}(k) = r \frac{\int_0^\infty \phi(\tau) \tilde{p}(k, \tau) d\tau}{\int_0^\infty \phi(\tau) d\tau}. \quad (50)$$

Of course, determining  $\tilde{p}_{st}(k)$  explicitly requires knowledge of the full time-dependent solution in the right hand side of (50). Nevertheless, it is clear that the integrals in the right hand side of (50) converge and that therefore there is a stationary state solution  $\tilde{p}_{st}(k)$  as  $t \rightarrow \infty$ .

**B. Power-law family:**  $\phi(\tau) \sim \tau^\alpha$  as  $\tau \rightarrow \infty$  with  $\alpha > -1$

Let us assume  $\phi(\tau) \sim \tau^\alpha$  as  $\tau \rightarrow \infty$  and that  $\alpha > -1$ . In this case, to analyse Eq. (7) at late times, we make the ansatz

$$\tilde{p}(k, t) \sim \frac{A}{t^\gamma} \quad \text{with} \quad 0 < \gamma < \alpha + 1. \quad (51)$$

Substituting this ansatz in Eq. (7), one finds that the lhs decays as  $t^{-\gamma-1}$  while the rhs decays as  $t^{-\gamma}$ . Thus the pre-factor of the term of order  $t^{-\gamma}$  in the rhs must be 0, which fixes the exponent  $\gamma$

$$\gamma = (\alpha + 1) \frac{D k^2}{r + D k^2}, \quad (52)$$

for any  $k$ . The above expression is consistent with the *a priori* assumption  $\gamma < \alpha + 1$ . Hence, as  $k \rightarrow 0$ ,

$$\tilde{p}(k, t) \sim A t^{-\gamma} = A \exp[-\gamma \ln(t)] \sim A \exp\left[-\frac{(\alpha + 1)D}{r} k^2 \ln(t)\right]. \quad (53)$$

The normalisation condition  $\tilde{p}(k = 0, t) = 1$  imposes  $A \rightarrow 1$  as  $k \rightarrow 0$ . Inverting the Fourier transform, one obtains a Gaussian distribution  $p(x, t) \sim \exp[-x^2/2\sigma^2(t)]$  where the variance grows with time logarithmically

$$\sigma^2(t) \approx \frac{2D}{r} (\alpha + 1) \ln(t). \quad (54)$$

In particular, for  $\alpha = 0$ , one recovers the result of Section (IV). Note that Gaussian diffusion with logarithmic variance as above is also observed in certain Markovian processes, for example, for Brownian motion when the diffusion coefficient decays as  $1/t$  at large times [27].

**C. The borderline case when  $\phi(\tau) \sim 1/\tau$  as  $\tau \rightarrow \infty$ , i.e.,  $\alpha = -1$**

In this borderline case, the following ansatz is appropriate

$$\tilde{p}(k, t) \sim \frac{A}{(\ln t)^\gamma} \quad (55)$$

where the exponent  $\gamma > 0$  is yet to be determined. Substituting this ansatz into the rhs of Eq. (7) and neglecting the lhs (which decays faster), one finds that

$$\gamma = \frac{D k^2}{r + D k^2}. \quad (56)$$

As above,  $A$  must tend to 1 when  $k \rightarrow 0$ . Hence, for small  $k$ ,

$$\tilde{p}(k, t) \sim \exp[-\beta \ln \ln t] \sim \exp\left[-\frac{D}{r} k^2 \ln \ln(t)\right], \quad (57)$$

indicating a Gaussian in real space with variance growing extremely slowly for large  $t$  as

$$\sigma^2(t) \approx \frac{2D}{r} \ln \ln(t). \quad (58)$$

**D. Stretched exponential family:**  $\phi(\tau) \sim \exp[a\tau^\beta]$  as  $\tau \rightarrow \infty$  and with  $a > 0$  and  $\beta > 0$

Here we consider a class of functions belonging to the family

$$\phi(\tau) \sim \exp[a\tau^\beta] \quad \text{where} \quad a > 0 \quad (59)$$

where the stretching exponent  $\beta$  is positive. In this model recent times ( $\tau$  close to  $t$ ) are thus sampled much more often than early times  $\tau \ll t$ . This form of memory decay is controlled with the parameter  $\beta$ , larger values of  $\beta$  corresponding to shorter memory ranges. We consider below three subclasses: (i)  $0 < \beta < 1$  (ii)  $\beta = 1$  and (iii)  $\beta > 1$ .

1. Case  $0 < \beta < 1$ :

In this case, the integral  $\int_0^t \phi(\tau) d\tau$  behaves for large  $t$  (to leading order) as

$$\int_0^t \phi(\tau) d\tau \sim \int_0^t e^{a\tau^\beta} d\tau \sim \frac{1}{a\beta} t^{1-\beta} e^{a t^\beta}. \quad (60)$$

Now, we make the ansatz that for large  $t$

$$\tilde{p}(k, t) \sim A \exp[-c t^\beta] \quad (61)$$

where we assume  $0 < c < a$  (to be justified *a posteriori*). We next substitute this ansatz in Eq. (7). The integral in the numerator of the second term on the rhs of Eq. (7) can be estimated, to leading order in large  $t$ , as (see Eq. (60))

$$\int_0^t \phi(\tau) \tilde{p}(k, \tau) d\tau \sim A \int_0^t e^{(a-c)\tau^\beta} d\tau \sim \frac{A}{(a-c)\beta} t^{1-\beta} e^{(a-c)t^\beta}. \quad (62)$$

Substituting these results in Eq. (7) gives

$$-c\beta t^{\beta-1} e^{-ct^\beta} \approx -(r + Dk^2) e^{-ct^\beta} + \frac{ra}{a-c} e^{-ct^\beta}. \quad (63)$$

For  $\beta < 1$ , the lhs decays faster than rhs and equating the rhs to 0 fixes the constant  $c$

$$c = a \frac{Dk^2}{r + Dk^2} < a \quad (64)$$

which is consistent with the *a priori* assumption  $c < a$ . Hence, this gives, for small  $k$ ,

$$\tilde{p}(k, t) \sim \exp\left[-\frac{aD}{r} k^2 t^\beta\right] \quad (65)$$

indicating once again a Gaussian distribution for  $p(x, t)$  with variance growing subdiffusively with time

$$\sigma^2(t) \approx \frac{2aD}{r} t^\beta \quad \text{where } 0 < \beta < 1. \quad (66)$$

2. Case  $\beta = 1$ :

We now consider the case,  $\phi(\tau) \sim e^{a\tau}$  as  $\tau \rightarrow \infty$ . In this case, the correct ansatz for  $\tilde{p}(k, t)$  turns out to be

$$\tilde{p}(k, t) \sim A e^{-bt} \quad (67)$$

where we assume  $b < a$  *a priori* (to be justified *a posteriori*). Substituting this ansatz in Eq. (7), we find that now the lhs and the rhs both scale as  $e^{-bt}$  for large  $t$ . Equating the two sides gives the identity

$$(r + Dk^2 - b) = r \frac{a}{a-b}. \quad (68)$$

This is a quadratic equation for  $b$  which can be solved to give

$$b_{\pm} = \frac{1}{2} \left[ r + Dk^2 + a \pm \sqrt{(r + Dk^2 + a)^2 - 4aDk^2} \right]. \quad (69)$$

It can be shown easily that  $b_- < a$ . The condition  $b < a$  thus forces us to choose  $b_-$ ,

$$b = b_- = \frac{1}{2} \left[ r + Dk^2 + a - \sqrt{(r + Dk^2 + a)^2 - 4aDk^2} \right]. \quad (70)$$

For  $k \rightarrow 0$ , we have

$$b \approx \frac{a}{r+a} Dk^2. \quad (71)$$

Hence, for small  $k$  and large  $t$ ,

$$\tilde{p}(k, t) \sim \exp \left[ -\frac{aD}{r+a} k^2 t \right]. \quad (72)$$

Once again,  $A \rightarrow 1$  at small  $k$  due to normalisation. Inverting the Fourier transform, one again gets a Gaussian distribution for  $p(x, t)$  with variance growing diffusively for large  $t$

$$\sigma^2(t) \approx \frac{2a}{r+a} D t. \quad (73)$$

Diffusion is normal in this case, but the effective diffusion coefficient of the process,  $aD/(r+a)$ , is smaller than the bare coefficient  $D$  for any positive resetting rate  $r$ .

### 3. Case $\beta > 1$ :

Finally, we consider the class of functions where  $\phi(\tau) \sim e^{a\tau^\beta}$  for large  $\tau$  with the exponent  $\beta > 1$ . In this case, the correct ansatz for  $\tilde{p}(k, t)$  turns out to be an exponential, i.e.,

$$\tilde{p}(k, t) \sim A e^{-bt} \quad (74)$$

where the constant  $b$  is yet to be determined. Substituting this ansatz in Eq. (7), we find that (as in the case  $\beta = 1$ ), the lhs and rhs both scale as  $\exp[-bt]$  at late times and equating them fixes the constant  $b$  simply as

$$b_1 = D k^2. \quad (75)$$

Note that  $b$  is now completely independent of the resetting rate  $r$ . Hence, in this case, one recovers *pure* diffusion (as in the  $r = 0$  case),

$$\tilde{p}(k, t) \sim A e^{-D k^2 t}. \quad (76)$$

When inverted, one obtains a Gaussian  $p(x, t)$  (the standard purely diffusive propagator) with variance growing linearly with  $t$  (independent of  $r$ )

$$\sigma^2(t) \approx 2 D t. \quad (77)$$

Thus, for this class of memory kernel  $\phi(\tau)$ , resetting is not effective at all, and the walker undergoes normal diffusion asymptotically at late times.

## VI. DISCUSSION

In this work we have considered diffusion with resetting to a position from the past chosen according to memory kernel (4). We have demonstrated that a remarkable range of long time behaviours for the variance of the distribution of the diffusive particle are made possible through different choices of the function  $\phi(\tau)$ , as summarised in Section V.

We recover previous results for diffusion with resetting [18] and the preferential relocation model [14] as limiting cases. For functions  $\phi(\tau)$  decaying faster than  $1/\tau$ , the distribution of the position converges towards a non-equilibrium stationary state at large times, whereas slower decays produce logarithmic diffusion. At the transition point between these two regimes, for  $\phi(\tau) \sim 1/\tau$ , we obtain a Gaussian with variance growing ultra-slowly as  $\ln \ln t$ , an unusual diffusive behaviour. When the memory function  $\phi$  *increases* with  $\tau$  as a stretched exponential with exponent  $\beta < 1$  (which mimics a form of memory decay since recent times are sampled much more frequently than early times) non-Markovian effects become weaker but the particle is still subdiffusive and characterised by a Gaussian distribution with variance  $t^\beta$ . This law is replaced by normal diffusion for  $\beta \geq 1$ .

Let us comment that although we have considered a one-dimensional process the results are easily generalisable to arbitrary spatial dimension. In the case where a stationary state exists we expect the form of the stationary state to depend on dimension as was the case in [28]. However in the case of a time-dependent scaling distribution we expect the scaling distribution to be a  $d$  dimensional Gaussian distribution with width growing in the same way as the one-dimensional case we have presented here. The reason being that when we generalise to  $d$  dimension the equation

(7) obeyed by the Fourier transform, it remains the same but with  $k$  replaced by  $|\underline{k}|$ , where the Fourier transform of  $p(\underline{x}, t)$  is now

$$\tilde{p}(\underline{k}, t) = \int_{-\infty}^{\infty} p(\underline{x}, t) e^{i\underline{k} \cdot \underline{x}} d^d x. \quad (78)$$

Therefore the small  $k$  expansions we have used in Section V will also apply in the dimension  $d$  case and we will obtain the same time dependences for the width.

We conclude by mentioning that the recurrence properties of random walks with memory and resetting are still poorly understood. Due to the non-Markovian nature of these processes, the study of their first passage times is a challenging problem [29].

**Acknowledgements** We acknowledge that some results of section IV were derived in the Master's Thesis of Tunrayo Adeleke-Larodo [30] and were also independently derived by Luca Giuggioli. We thank them for useful discussions.

MRE would like to acknowledge funding from EPSRC under grant number EP/J007404/1. SNM thanks the Higgs Centre for hospitality during the writing of the manuscript. DB acknowledges support from PAPIIT (UNAM) grant IN105015.

- 
- [1] Chandrasekhar S 1943 Stochastic problems in physics and astronomy, *Rev. Mod. Phys.* **15**, 1
  - [2] Berg H C 1993 *Random walks in Biology*, Princeton University Press.
  - [3] Grady L 2006 Random walks for image segmentation, *IEEE T. Pattern Anal.* **28**, 1768
  - [4] Bouchaud J-P and Georges A 1990 Anomalous diffusion in disordered media: Statistical mechanisms, models and physical applications, *Phys. Rep.* **195**, 127
  - [5] Metzler R, Klafter J 2004 The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics *J. Phys. A: Math. Theor.*, **37**, R161
  - [6] Montroll, E W and Weiss, G-H 1965 Random Walks on Lattices. II *J. Math. Phys.* **6**, 167
  - [7] Schütz G M and Trimper S 2004 Elephants can always remember: exact long-range memory effects in a non-Markovian random walk, *Phys. Rev. E* **70**, 045101 (R)
  - [8] Zerner M P W 2005 Multi-excited random walks on integers, *Probab. Theory Relat. Fields* **133**, 98
  - [9] Basdevant A L and Singh A 2008 On the speed of a cookie random walk, *Probab. Theory Relat. Fields* **141**, 625
  - [10] Davis B 1990 Reinforced random walk, *Probab. Th. Rel. Fields* **84**, 203
  - [11] Foster J G, Grassberger P and Paczuski M 2009 Reinforced walks in two and three dimensions, *New J. Phys.* **11**, 023009.
  - [12] Gaustestad A O and Mysterud I 2005 Intrinsic scaling complexity in animal dispersion and abundance, *Am. Nat.* **165**, 44
  - [13] Gaustestad A O and Mysterud I 2006 Complex animal distribution and abundance from memory-dependent kinetics, *Ecol. Complex.* **3**, 44
  - [14] Boyer D and Solis-Salas C 2014 Random walks with preferential relocations to places visited in the past and their application to biology, *Phys. Rev. Lett.* **112**, 240601
  - [15] Boyer D and Romo-Cruz J C R 2014 Solvable random walk model with memory and its relations with Markovian models of anomalous diffusion, *Phys. Rev. E* **90**, 042136
  - [16] Boyer D and Pineda I 2016 Slow Lévy flights, *Phys. Rev. E* **93**, 022103
  - [17] Manrubia S C and Zanette D H 1999 Stochastic multiplicative processes with reset events, *Phys. Rev. E* **59**, 4945
  - [18] Evans M R and Majumdar S N 2011 Diffusion with Stochastic Resetting, *Phys. Rev. Lett.* **106**, 160601
  - [19] Majumdar S N, Sabhapandit S and Schehr G 2015 Dynamical transition in the temporal relaxation of stochastic processes under resetting, *Phys. Rev. E* **91**, 052131
  - [20] Mendez V and Campos D 2016 Characterization of stationary states in random walks with resetting, *Phys. Rev. E* **93**, 022106
  - [21] Eule S and Metzger J J 2016 Non-equilibrium steady states of stochastic processes with intermittent resetting, *New J. Phys.* **18**, 033006
  - [22] Pal A, Kundu A and Evans M R 2016 Diffusion under time-dependent resetting, *J. Phys. A: Math. Theor.* **49**, 225001
  - [23] Nagar A and Gupta S 2016 Diffusion with stochastic resetting at power-law times, *Phys. Rev. E* **93** 060102(R)
  - [24] Majumdar S N, Sabhapandit S and Schehr G 2015 Random walk with random resetting to the maximum position, *Phys. Rev. E* **92**, 052126
  - [25] Song C, Koren T, Wang P and Barabási A L 2010 Modelling the scaling properties of human mobility, *Nature Phys.* **6**, 818
  - [26] Abramowitz M and Stegun I A 2012 *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*. Courier Dover Publications, (2012).
  - [27] Bodrova A S, Chechkin A V, Cherstvy A G and Metzler R 2015 Ultraslow scaled Brownian motion, *New J. Phys.* **17**, 063038

- [28] Evans M R and Majumdar S N 2014 Diffusion with resetting in arbitrary spatial dimension, *J. Phys. A: Math . Theor.* **47**, 285001
- [29] Bray A J, Majumdar S N and Schehr G 2013 Persistence and first-passage properties in nonequilibrium systems *Adv. Phys.* **62**, 225
- [30] Adeleke-Larodo T 2015 *Master's Thesis*, University of Edinburgh